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# The Magnetization–Energy Scaling Limit in High Dimension

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In the single-phase region (including the critical point) of a nearest-neighbor Ising ferromagnet with zero external field, the block magnetization and energy within the infinite-volume system are, asymptotically for large block size, independent Gaussian variables when the dimension $d$ exceeds four. For other models, including ones with long-range interactions, a sufficient condition for such triviality of the scaling limit is finiteness of the "bubble quantity."

**KEY WORDS:** Scaling limit; triviality; Ising model; $\phi^4$ field theory; upper critical dimension; bubble diagram; block variables; renormalization group.

## 1. INTRODUCTION

One of the major features of the renormalization group approach to critical phenomena is that the scaling limit at an ordinary $d$-dimensional ferromagnetic critical point should be Gaussian for $d \geq 4$.(34) This phenomenon has been substantially confirmed by the rigorous analyses of Aizenman,(1,2) Fröhlich,(12) Aizenman and Graham,(5) and Gawedzki and Kupiainen(18,19) (for an extensive review, see ref. 13). Our purpose here is to sharpen some of the previously existing rigorous results for $d > 4$ in two ways. First, we eliminate extraneous (and unverified) hypotheses on the decay or regularity properties of the critical two-point function; such hypotheses were previously needed for $d \leq 6$.(1) Second, we go beyond a single-variable block magnetization analysis to obtain bivariate magnetization–energy results. A bivariate block variable analysis oriented toward

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critical scaling limits was performed in ref. 10; here we carry the ideas of that paper somewhat farther.

We mainly restrict our attention to the standard Ising model. Extensions of our results to such systems as lattice $\phi^4$ and non-nearest-neighbor models (including long-range models with $d<4$) will be discussed briefly in Section 5. For both standard Ising and more general models, the criterion which determines the upper critical dimension, as in refs. 1 and 31, is the finiteness of the the "bubble quantity" $\sum_x \langle \phi_0 \phi_x \rangle^2$ at the critical point. An interesting open problem is to prove that the $d=4$ scaling limit is Gaussian. Our analysis does not resolve this issue even though Gaussianness of the scaling limit at the critical point can sometimes be proved without requiring finiteness of the bubble quantity—e.g., if for some function $g(r)$ which is decreasing to zero as $r \to \infty$, $\langle \phi_0 \phi_x \rangle / (g(\|x\|) \, \|x\|^{-d/2})$ is bounded away from zero and infinity as $\|x\| \to \infty$. (See the remark at the end of Section 3.)

## 2. RESULTS FOR THE STANDARD ISING MODEL

Let $\{\sigma_x(\beta): x \in \mathbb{Z}^d\}$ denote the $\pm 1$-valued spin random variables of a $d$-dimensional nearest-neighbor spin-1/2 Ising ferromagnet at inverse temperature $\beta$ and zero external field obtained as the infinite-volume limit of finite-volume systems with free boundary conditions. For the cubic block, $\Lambda_n = \{-n, -n+1, \ldots, n\}^d$, the block magnetization and energy are

$$M_n(\beta) = \sum_{x \in \Lambda_n} \sigma_x(\beta) \tag{2.1}$$

and

$$E_n(\beta) = \sum_{x \in \Lambda_n} \left[ -\frac{1}{2} \sum_{y \in \mathbb{Z}^d} J_{xy} \sigma_x(\beta) \, \sigma_y(\beta) \right] \tag{2.2}$$

where

$$J_{xy} = \begin{cases} 1 & \text{if} \quad \|x-y\| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

and $\|\cdot\|$ denotes the Euclidean length in $\mathbb{Z}^d$.

Let $\beta_c = \beta_c(d)$ denote the inverse critical temperature (i.e., the infimum of $\beta$ such that the spontaneous magnetization at $\beta$ is nonzero). For $d>4$ (in fact, for $d>7/2$) it is known[3] that there is a unique infinite-volume Gibbs distribution both for $\beta < \beta_c$ and $\beta = \beta_c$. Denoting expectations by $\langle \cdot \rangle$, we further define

$$K_n(\beta) = \langle M_n(\beta)^2 \rangle \tag{2.4}$$

$$L_n(\beta) = \langle [E_n(\beta) - \langle E_n(\beta) \rangle]^2 \rangle \tag{2.5}$$

We can now state our main result concerning the asymptotic behavior of $M_n(\beta_n)$ and $E_n(\beta_n)$ for a sequence $\beta_n$. The cases of main interest are when $\beta_n = \beta_c$ for all $n$, or $\beta_n$ converges to $\beta_c$.

**Theorem 1.** Let $d > 4$ be fixed; let $\beta_n$ be any sequence of inverse temperatures in $[0, \beta_c]$ and define

$$X_n = [K_n(\beta_n)]^{-1/2} M_n(\beta_n) \tag{2.6}$$

$$Y_n = [L_n(\beta_n)]^{-1/2} [E_n(\beta_n) - \langle E_n(\beta_n) \rangle] \tag{2.7}$$

Then $(X_n, Y_n)$ converges in distribution as $n \to \infty$ to a pair of independent standard Gaussian random variables; i.e., for any bounded, continuous, complex-valued function $g$ on $\mathbb{R}^2$,

$$\lim_{n \to \infty} \langle g(X_n, Y_n) \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) \frac{1}{2\pi} e^{-(x^2 + y^2)/2} \, dx \, dy \tag{2.8}$$

The proof of Theorem 1 will be given in the next two sections. In Section 3, we show that $X_n$ alone converges to a standard Gaussian variable; then in Section 4, we show that $Y_n$ does likewise and that $Y_n$ is asymptotically independent of $X_n$. In Section 5 we discuss the extension of Theorem 1 to more general models.

*Remark.* The proof of Theorem 1 (and its extensions given in Section 5) can easily be modified to apply to the lattice of block variables $X(n, x)$ and $Y(n, x)$ defined for $x \in \mathbb{Z}^d$ exactly like $X_n$ and $Y_n$ except that $\Lambda_n$ is replaced by $(2n+1)x + \Lambda_n$. Considering the joint distribution of these variables for finitely many $x$'s, one always has tightness (i.e., every subsequence has convergent subsubsequences), but not, in general, convergence; for example, a subsequence with $\beta_n$ bounded away from $\beta_c$ would have its $X(n, x)$'s asymptotically independent for different blocks, while this should not be the case for a subsequence with, say, $\beta_n = \beta_c$. One can show, however, that every limit in distribution is jointly Gaussian and (without needing subsequences) that the $Y(n, x)$'s are asymptotically independent of each other and of the $X(n, x)$'s.

## 3. BLOCK MAGNETIZATION

The following proposition immediately implies that $X_n$ converges to a standard Gaussian random variable.

**Proposition 2.** Under the assumptions of Theorem 1,

$$\lim_{n \to \infty} \langle \exp(tX_n) \rangle = \exp(t^2/2), \qquad t \in \mathbb{R} \tag{3.1}$$

*Proof*. From the Lee–Yang result[25] that the zeros in the complex plane of $\langle \exp(zX_n) \rangle$ are all pure imaginary, it follows that[28]

$$\exp(t^2/2 + U_{4,n} t^4/4!) \leqslant \langle \exp(tX_n) \rangle \leqslant \exp(t^2/2), \qquad t \in \mathbb{R} \tag{3.2}$$

where $U_{4,n}$ (which is necessarily negative) is defined by

$$\begin{aligned} U_{4,n} &\equiv E(X_n^4) - 3[E(X_n^2)]^2 \\ &= [K_n(\beta_n)]^{-2} \sum_{x_1, x_2, x_3, x_4 \in \Lambda_n} [\langle \sigma_{x_1}(\beta_n) \, \sigma_{x_2}(\beta_n) \, \sigma_{x_3}(\beta_n) \, \sigma_{x_4}(\beta_n) \rangle \\ &\quad - \langle \sigma_{x_1}(\beta_n) \, \sigma_{x_2}(\beta_n) \rangle \langle \sigma_{x_3}(\beta_n) \, \sigma_{x_4}(\beta_n) \rangle \\ &\quad - \langle \sigma_{x_1}(\beta_n) \, \sigma_{x_3}(\beta_n) \rangle \langle \sigma_{x_2}(\beta_n) \, \sigma_{x_4}(\beta_n) \rangle \\ &\quad - \langle \sigma_{x_1}(\beta_n) \, \sigma_{x_4}(\beta_n) \rangle \langle \sigma_{x_2}(\beta_n) \, \sigma_{x_3}(\beta_n) \rangle] \end{aligned} \tag{3.3}$$

To obtain (3.1), it then suffices to prove that $U_{4,n} \to 0$ as $n \to \infty$.

Denoting the summand in (3.3) by $u_{4,n}(x_1, x_2, x_3, x_4)$, we will use Aizenman's inequality,[2]

$$\begin{aligned} -u_{4,n}(x_1, x_2, x_3, x_4) \leqslant 2 \sum_{y \in \mathbb{Z}^d} & G_n(x_1 - y) \, G_n(x_2 - y) \\ & \times G_n(x_3 - y) \, G_n(x_4 - y) \end{aligned} \tag{3.4}$$

where

$$G_n(x) = \langle \sigma_y(\beta_n) \, \sigma_{y+x}(\beta_n) \rangle \geqslant 0 \tag{3.5}$$

Thus

$$0 \leqslant -U_{4,n} \leqslant 2[K_n(\beta_n)]^{-2} \sum_{y \in \mathbb{Z}^d} \left[ \sum_{x \in \Lambda_n} G_n(x - y) \right]^4 \tag{3.6}$$

We next claim that for any $y \in \mathbb{Z}^d$,

$$\sum_{x \in \Lambda_n} G_n(x - y) \leqslant 2^d \, |\Lambda_n|^{-1} \, K_n(\beta_n) \tag{3.7}$$

where $|\Lambda_n| = (2n+1)^d$ is the volume of $\Lambda_n$; then (3.6) yields

$$\begin{aligned} 0 \leqslant -U_{4,n} &\leqslant 2^{2d+1} \, |\Lambda_n|^{-2} \sum_{y \in \mathbb{Z}^d} \left[ \sum_{x \in \Lambda_n} G_n(x - y) \right]^2 \\ &= 2^{2d+1} \, |\Lambda_n|^{-2} \sum_{x_1, x_2 \in \Lambda_n} \sum_{y \in \mathbb{Z}^d} G_n(x_1 - y) \, G_n(x_2 - y) \end{aligned} \tag{3.8}$$

To obtain (3.7), we use the fact[26,30] that $G_n((x^1,..., x^d))$ is a decreasing function of each $|x^j|$ (with $x^i$ fixed for $i \neq j$) along with the positivity of $G_n$ and its invariance under $x^j \to -x^j$:

$$\begin{aligned}\sum_{x \in \Lambda_n} G_n(x-y) &\leqslant \sum_{x \in \Lambda_n} G_n(x) \\ &\leqslant 2^d \sum_{x \in \{0,1,...,n\}^d} G_n(x) \\ &\leqslant 2^d\, |\Lambda_n|^{-1} \sum_{x_1 \in \Lambda_n} \sum_{x_2 \in \Lambda_n} G_n(x_2 - x_1) \\ &= 2^d\, |\Lambda_n|^{-1}\, K_n(\beta_n) \end{aligned} \tag{3.9}$$

[Another proof of (3.7) will be given in Section 5.]

Now by the monotonicity of $\langle \sigma_x(\beta)\, \sigma_{x'}(\beta) \rangle$ in $\beta$ and the assumption that $\beta_n \leqslant \beta_c$, we have

$$0 \leqslant G_n(x) \leqslant G_c(x) \equiv \langle \sigma_y(\beta_c)\, \sigma_{y+x}(\beta_c) \rangle \tag{3.10}$$

so that by (3.8),

$$\limsup_{n \to \infty} (-U_{4,n}) \leqslant 2^{2d+1} \limsup_{|x| \to \infty} \sum_{y \in \mathbb{Z}^d} G_c(x-y)\, G_c(y) \tag{3.11}$$

By the infrared bounds of ref. 14, it is known that for $d > 4$, the "bubble quantity" is finite at the critical point:

$$\sum_{x \in \mathbb{Z}^d} G_c(x)^2 < \infty \tag{3.12}$$

Since we thus have

$$G_c(x) = (2\pi)^{-d/2} \int_{[-\pi,\pi]^d} e^{-i(k,x)} \hat{G}_c(k)\, dk \tag{3.13}$$

for some $\hat{G}_c \in L^2([-\pi, \pi]^d, dk)$, where $(k, x) = k^1 x^1 + \cdots + k^d x^d$, we obtain

$$\sum_{y \in \mathbb{Z}^d} G_c(x-y)\, G_c(y) = \int_{[-\pi,\pi]^d} e^{-i(k,x)} [\hat{G}_c(k)]^2\, dk \tag{3.14}$$

The rhs of (3.11) then vanishes by the Riemann–Lebesgue lemma and $U_{4,n} \to 0$, as desired, completing the proof of Proposition 2.

*Remark.* We consider whether the above proof can be revised to cover cases, such as $d = 4$, where the bubble quantity of (3.12) is logarithmically divergent. With $\beta_n = \beta_c$ for all $n$, a proof that $X_n$ has a Gaussian limit based on Aizenman's inequality (3.4) requires (see (3.6)) that

$$\lim_{n \to \infty} [K_n(\beta_c)]^{-2} H_n = 0 \tag{3.15}$$

where

$$H_n = \sum_{y \in \mathbb{Z}^d} \left[ \sum_{x \in \Lambda_n} G_c(x-y) \right]^4 \tag{3.16}$$

When $d=4$, $\|x\|^{d-2} G_c(x)$ is bounded away from infinity,[32] and is believed to be bounded away from zero (with no "logarithmic" corrections; see, e.g., ref. 9) as $\|x\| \to \infty$. Such behavior of $G_c$ would imply (after bounding sums by integrals and then scaling all variables by $n$) that $H_n \geqslant cn^{d+8}$, while $K_n(\beta_c) \leqslant c'n^{d+2}$, for finite positive $c$ and $c'$ so that $[K_n(\beta_c)]^{-2} H_n$ would not tend to zero when $d=4$. Nevertheless, it is interesting to note that if $\|x\|^{d/2} G_c(x) \to 0$ as $\|x\| \to \infty$, no matter how slowly, one can prove (assuming some extra properties of $G_c$) that $[K_n(\beta_c)]^{-2} H_n$ does tend to zero. For example, suppose

$$\text{const. } g(\|x\|)/\|x\|^{d/2} \leqslant G_c(x) \leqslant \text{const.}' \, g(\|x\|)/\|x\|^{d/2} \tag{3.17}$$

for some decreasing function $g$ with $g(r) \to 0$ as $r \to \infty$. Then $K_n(\beta_c)$ is bounded above and below by some constant times

$$n^d \int_0^n r^{-d/2} g(r) \, r^{d-1} \, dr \tag{3.18}$$

while $H_n$ is bounded above by some constant times

$$n^d [n^{-d} K_n(\beta_c)]^4 + \sum_{y \notin \Lambda_{2n}} \left[ \sum_{x \in \Lambda_n} g(\|x-y\|) \, \|x-y\|^{-d/2} \right]^4 \tag{3.19}$$

However

$$[K_n(\beta_c)]^{-2} n^d [n^{-d} K_n(\beta_c)]^4 \leqslant c' \left[ n^{-d/2} \int_0^n r^{d/2-1} g(r) \, dr \right]^2 \tag{3.20}$$

which tends to zero as $n \to \infty$ because $g(r) \to 0$ as $r \to \infty$. Furthermore, because $g$ is decreasing, the second term of (3.19) is bounded by some constant times

$$[g(n)]^4 \int_{2n}^{\infty} (n^d r^{-d/2})^4 \, r^{d-1} \, dr = [g(n)]^4 \, n^{4d} (2n)^{-d}/d \tag{3.21}$$

Thus, because $g$ is decreasing to zero, we have

$$\begin{aligned}\lim_{n\to\infty} [K_n(\beta_c)]^{-2} H_n &\leqslant \lim_{n\to\infty} c' \frac{[g(n)]^4}{[n^{-d/2}\int_0^n r^{d/2-1}g(r)]^2} \\ &\leqslant \lim_{n\to\infty} c'(2/d)[g(n)]^4/[g(n)]^2 = 0\end{aligned} \tag{3.22}$$

## 4. BLOCK ENERGY

The following theorem implies that $Y_n$ converges to a standard Gaussian variable.

**Proposition 3.** Under the assumptions of Theorem 1,

$$\lim_{n\to\infty} \langle \exp(sY_n)\rangle = \exp(s^2/2), \qquad s \geqslant 0 \tag{4.1}$$

*Proof.* By considering subsequences, we may assume, without loss of generality, that $\beta_n \to \bar{\beta} \in [0, \beta_c]$. We introduce the following notation:

$$\langle A\rangle_s = \langle A \exp(sY_n)\rangle / \langle \exp(sY_n)\rangle \tag{4.2}$$

$$\langle A; B\rangle_s = \langle AB\rangle_s - \langle A\rangle_s \langle B\rangle_s, \qquad \langle A; B\rangle = \langle A; B\rangle_0 \tag{4.3}$$

By Taylor's formula with remainder,

$$\begin{aligned}\log\langle \exp(sY_n)\rangle &= \frac{s^2}{2} L_n(\beta_n)^{-2} \langle E_n(\beta_n); E_n(\beta_n)\rangle_{\theta s} \\ &= \frac{s^2}{2} \frac{\langle E_n(\beta_n); E_n(\beta_n)\rangle_{\theta s}}{\langle E_n(\beta_n); E_n(\beta_n)\rangle_0}\end{aligned} \tag{4.4}$$

where $\theta = \theta(n, s) \in [0, 1]$. To obtain (4.1) from (4.4), it suffices to prove that for any sequence $s_n \in [0, s]$,

$$\lim_{n\to\infty} |\Lambda_n|^{-1} \langle E_n(\beta_n); E_n(\beta_n)\rangle_{s_n} = \sum_{x\in\mathbb{Z}^d} \langle \varepsilon_0(\bar{\beta}); \varepsilon_x(\bar{\beta})\rangle_0 < \infty \tag{4.5}$$

where $\varepsilon_x(\beta)$ is the local energy variable,

$$\varepsilon_x(\beta) = -\frac{1}{2} \sum_{y\in\mathbb{Z}^d} J_{xy} \sigma_x(\beta)\, \sigma_y(\beta) \tag{4.6}$$

Now $\langle \varepsilon_x(\beta); \varepsilon_y(\beta)\rangle_s \geqslant 0$,[15,21] and

$$\langle E_n(\beta); E_n(\beta)\rangle_s = \sum_{x, x_2\in\Lambda_n} \langle \varepsilon_{x_1}(\beta); \varepsilon_{x_2}(\beta)\rangle_s \tag{4.7}$$

so that by the dominated convergence theorem and standard arguments, it suffices to have

$$\lim_{n\to\infty} \langle \varepsilon_{x_1}(\beta_n); \varepsilon_{x_2}(\beta_n)\rangle_{s_n} = \langle \varepsilon_{x_1}(\bar\beta); \varepsilon_{x_2}(\bar\beta)\rangle_0 \tag{4.8}$$

and

$$\sum_{x\in\mathbb{Z}^d} \sup_{0\leqslant\beta'\leqslant\beta_c, 0\leqslant s'\leqslant s} \langle \varepsilon_0(\beta'); \varepsilon_x(\beta')\rangle_{s'} < \infty \tag{4.9}$$

in order to obtain (4.5) and complete the proof.

For (4.8), it is enough to have

$$\lim_{n\to\infty} \langle \sigma_{x_1}(\beta_n)\cdots\sigma_{x_{2n}}(\beta_n)\rangle_{s_n} = \langle \sigma_{x_1}(\bar\beta)\cdots\sigma_{x_{2n}}(\bar\beta)\rangle \tag{4.10}$$

at least for $n=1$ and 2. Now by the GKS inequalities[15,21]

$$\begin{aligned}\langle \sigma_{x_1}(\tilde\beta_n)\cdots\sigma_{x_{2n}}(\tilde\beta_n)\rangle &\leqslant \langle \sigma_{x_1}(\beta_n)\cdots\sigma_{x_{2n}}(\beta_n)\rangle_{s_n}\\ &\leqslant \langle \sigma_{x_1}(\beta_n)\cdots\sigma_{x_{2n}}(\beta_n)\rangle\end{aligned} \tag{4.11}$$

where

$$\tilde\beta_n = \beta_n - s_n/[L_n(\beta_n)]^{1/2} \to \bar\beta \qquad \text{as} \qquad n\to\infty \tag{4.12}$$

since

$$\begin{aligned} L_n(\beta_n) &\geqslant \frac{1}{4}\sum_{x\in\Lambda_n}\sum_{y\in\mathbb{Z}^d} J_{xy}^2 \langle \sigma_x(\beta_n)\,\sigma_y(\beta_n); \sigma_x(\beta_n)\,\sigma_y(\beta_n)\rangle\\ &= \frac{d}{2}\,|\Lambda_n|\,[1-\langle \sigma_0(\beta_n)\,\sigma_{0'}(\beta_n)\rangle^2]\\ &\geqslant \frac{d}{2}\,|\Lambda_n|\,[1-\langle \sigma_0(\beta_c)\,\sigma_{0'}(\beta_c)\rangle^2] \to \infty \qquad \text{as} \quad n\to\infty \end{aligned} \tag{4.13}$$

where $0'$ denotes a nearest neighbor of the origin 0. Hence (4.10) follows from the continuity of $\langle \sigma_{x_1}(\beta)\cdots\sigma_{x_{2n}}(\beta)\rangle$ as a function of $\beta\in[0,\beta_c]$. The continuity was proved by Lebowitz[23,24] to be a consequence of the vanishing of the spontaneous magnetization which is valid by definition for $\beta<\beta_c$ and is also valid at $\beta=\beta_c$ (at least for $d>7/2$).[3] (Another argument for continuity will be given in Section 5.)

The quantity appearing in (4.9) is basically a specific heat whose finiteness for $d>4$ was proved by Sokal.[31] For completeness (and for the purpose of Section 5) we provide a proof.

By the "Lebowitz inequality" for $u_{4,n}$,[17,22]

$$\begin{aligned}\langle \varepsilon_0(\beta'); \varepsilon_x(\beta')\rangle_{s'} &= \frac{1}{4} \sum_{y,\, y' \in \mathbb{Z}^d} J_{0y} J_{xy'} \langle \sigma_0(\beta')\, \sigma_y(\beta'); \sigma_x(\beta')\, \sigma_{y'}(\beta')\rangle_{s'} \\ &\leqslant \frac{1}{4} \sum_{y,\, y' \in \mathbb{Z}^d} J_{0y} J_{xy'} [\langle \sigma_0(\beta')\, \sigma_x(\beta')\rangle_{s'} \langle \sigma_y(\beta')\, \sigma_{y'}(\beta')\rangle_{s'} \\ &\quad + \langle \sigma_0(\beta')\, \sigma_{y'}(\beta')\rangle_{s'} \langle \sigma_y(\beta')\, \sigma_x(\beta')\rangle_{s'}] \end{aligned} \tag{4.14}$$

Then, by the GKS inequalities, one gets (4.9) provided that

$$\sum_{x \in \mathbb{Z}^d} \sum_{y,\, y' \in \mathbb{Z}^d} J_{0y} J_{xy'} G_c(x)\, G_c(y' - y) < \infty \tag{4.15}$$

This follows from the finiteness of the bubble quantity (3.12), as can be seen by various arguments. One argument uses only (3.12) and the fact that $J_{xy} = J_{y-x}$ with $\sum_{x \in \mathbb{Z}^d} |J_x| < \infty$; it begins by expressing the quantity in (4.15) as a fourfold convolution evaluated at the origin (we leave the details to the reader).

We conclude this section of the paper with a result which yields Theorem 1.

**Proposition 4.** Under the assumptions of Theorem 1,

$$\lim_{n \to \infty} \langle \exp(tX_n + sY_n)\rangle = \exp\left(\frac{t^2 + s^2}{2}\right) \quad \text{for } t \in \mathbb{R} \text{ and } s \geqslant 0 \tag{4.16}$$

Hence $(X_n, Y_n)$ converges in distribution as $n \to \infty$ to a pair of independent standard Gaussian random variables.

*Proof.* First we show that the last statement of the proposition follows from (4.16). Using the bound

$$|\langle \exp(zX_n + wY_n)\rangle| \leqslant \langle \exp([\operatorname{Re} z]\, X_n + [\operatorname{Re} w]\, Y_n)\rangle, \quad z, w \in \mathbb{C} \tag{4.17}$$

and standard complex variable arguments, we see that (4.16) implies

$$\lim_{n \to \infty} \langle \exp(zX_n + wY_n)\rangle = \exp\left(\frac{z^2 + w^2}{2}\right) \quad \text{for} \quad z \in \mathbb{C} \text{ and } \operatorname{Re} w > 0 \tag{4.18}$$

Taking $w = \alpha + iv$ and $z = iu$ with $\alpha > 0$ and denoting by $\rho_n(dx, dy)$ the joint distribution of $(X_n, Y_n)$, we see from (4.18) that

$$\int_{\mathbb{R}^2} f(x, y)\, e^{\alpha y}\, \rho_n(dx, dy) \to \int_{\mathbb{R}^2} f(x, y)\, e^{\alpha y} (2\pi)^{-1}\, e^{-(x^2 + y^2)/2}\, dx\, dy \tag{4.19}$$

for any bounded, continuous function $f$ of compact support. But this implies the same with $f(x, y)\, e^{\alpha y}$ replaced by any bounded, continuous function $g(x, y)$ of compact support, which in turn implies (2.8) as desired, since $\rho_n(dx, dy)$ has total mass one.

Now, to obtain (4.16), it suffices by Proposition 3 to show that

$$\lim_{n\to\infty} \langle \exp(tX_n)\rangle_s = \exp(t^2/2) \tag{4.20}$$

Since, by either Gaussian correlation inequalities[29] or the Lee–Yang property,[28]

$$\langle \exp(tX_n)\rangle_s \leqslant \exp\left(\frac{t^2}{2}\langle X_n^2\rangle_s\right) \tag{4.21}$$

it suffices, by Theorem 2 and standard complex variable arguments, to prove that for each $k = 1, 2,\ldots$

$$\lim_{n\to\infty} (\langle X_n^{2k}\rangle - \langle X_n^{2k}\rangle_s) = 0 \tag{4.22}$$

To show this, we write

$$\langle X_n^{2k}\rangle - \langle X_n^{2k}\rangle_s = [K_n(\beta_n)]^{-k} \int_0^s \left(-\frac{d}{dr}\langle M_n^{2k}\rangle_r\right) dr \tag{4.23}$$

and use the identity

$$\begin{aligned} -\frac{d}{dr}&\langle M_n^{2k}\rangle_r \\ &= \langle M_n^{2k}; -Y_n\rangle_r \\ &= \frac{1}{2}[L_n(\beta_n)]^{-1/2} \\ &\quad \times \sum_{x_1,\ldots,x_{2k}\in\Lambda_n} \sum_{x\in\Lambda_n} \sum_{y\in\mathbb{Z}^d} J_{xy} \langle \sigma_{x_1}(\beta_n)\cdots\sigma_{x_{2k}}(\beta_n); \sigma_x(\beta_n)\,\sigma_y(\beta_n)\rangle_r \end{aligned}$$

Next we use the following inequalities of ref. 29 (where we write $\langle x_1\cdots x_m\rangle$ to denote $\langle \sigma_{x_1}(\beta_n)\cdots\sigma_{x_m}(\beta_n)\rangle_r$):

$$\begin{aligned} \langle x_1\cdots x_{2k}xy\rangle &- \langle x_1\cdots x_{2k}\rangle\langle xy\rangle \\ &\leqslant \sum_{\substack{i,j=1\\ i\neq j}}^{2k} \langle x_i x\rangle\langle x_j y\rangle \sum{}' \langle x_{i_1}x_{i_2}\rangle\cdots\langle x_{i_{2k-3}}x_{i_{2k-2}}\rangle \end{aligned} \tag{4.24}$$

where $\sum'$ is a sum over all pairings of $\{1,..., 2k\}\backslash\{i, j\}$ into pairs $\{i_1, i_2\}$, $\{i_3, i_4\}$,.... Combining these inequalities with the GKS inequalities[15,21], we have

$$\begin{aligned} 0 &\leqslant -\frac{d}{dr}\langle M_n^{2k}\rangle_r \\ &\leqslant C_k[L_n(\beta_n)]^{-1/2}[K_n(\beta_n)]^k \\ &\quad\times\left(\sum_{y\in\mathbb{Z}^d} J_{xy}\right)\sup_{y\in\mathbb{Z}^d}\sum_{x'\in\Lambda_n}\langle\sigma_{x'}(\beta_n)\,\sigma_y(\beta_n)\rangle \end{aligned} \tag{4.25}$$

where $C_k$ is a combinatorial coefficient depending only on $k$ and where we have used that

$$\sum_{y_1, y_2\in\Lambda_n}\langle\sigma_{y_1}(\beta_n)\,\sigma_{y_2}(\beta_n)\rangle_r \leqslant \sum_{y_1, y_2\in\Lambda_n}\langle\sigma_{y_1}(\beta_n)\,\sigma_{y_2}(\beta_n)\rangle = K_n(\beta_n) \tag{4.26}$$

The last summation in (4.25) may be bounded using (3.7) to obtain

$$0\leqslant -\frac{d}{dr}\langle M_n^{2k}\rangle_r \leqslant C_{k,d}[L_n(\beta_n)]^{-1/2}[K_n(\beta_n)]^{k+1}|\Lambda_n|^{-1} \tag{4.27}$$

so that

$$0\leqslant\langle X_n^{2k}\rangle - \langle X_n^{2k}\rangle_s \leqslant C_{k,d}s\,|\Lambda_n|^{-1}[L_n(\beta_n)]^{-1/2}K_n(\beta_n) \tag{4.28}$$

By (4.13), $L_n(\beta_n)\geqslant \text{const}\cdot|\Lambda_n|\geqslant\text{const}'\cdot|\Lambda_{2n}|$, while

$$K_n(\beta_n)\leqslant K_n(\beta_c)\leqslant|\Lambda_n|\sum_{x\in\Lambda_{2n}} G_c(x) \tag{4.29}$$

Thus, to obtain (4.22) and complete the proof, it suffices to show that

$$\sum_{x\in\Lambda_{2n}}|\Lambda_{2n}|^{-1/2}G_c(x)\to 0 \qquad \text{as } n\to\infty \tag{4.30}$$

For $d>4$, the finiteness of the bubble quantity means that $G_c(\cdot)\in l^2(\mathbb{Z}^d)$ and the left-hand expression of (4.30) is the inner product of $G_c(\cdot)$ with a vector in $l^2(\mathbb{Z}^d)$ tending weakly to zero as $n\to\infty$; hence (4.30) is valid for $d>4$.

*Remark.* The arguments just used to prove Proposition 4 can be slightly modified for use in showing asymptotic independence of $X_n$ and $Y_n$ even when the bubble quantity is infinite. For example, suppose $d=4$ and $\beta_n\to\beta_c$. Then we may use the infrared bounds of ref. 32,

$$G_c(x)\leqslant\text{const}\cdot|x|^{2-d} = \text{const}\cdot|x|^{-2} \tag{4.31}$$

together with (4.28) to obtain

$$0 \leqslant \langle X_n^{2k} \rangle - \langle X_n^{2k} \rangle_s \leqslant \text{const} \cdot s \left[ \frac{|\Lambda_n|}{L_n(\beta_n)} \right]^{1/2} \tag{4.32}$$

But by arguments like those used in (4.7)–(4.10) and (4.13),

$$\liminf_{n \to \infty} \frac{L_n(\beta_n)}{|\Lambda_n|} \geqslant \sum_{x \in \mathbb{Z}^d} \langle \varepsilon_0(\beta_c); \varepsilon_x(\beta_c) \rangle \tag{4.33}$$

The right-hand side of (4.33) is a specific heat which, as discussed above, is finite for $d > 4$,[31] but which is predicted to be infinite for $d = 4$ by renormalization group calculations (see, e.g., ref. 9). For certain $\phi^4$ models, it can presumably be proved infinite by the methods of ref. 19.

## 5. RESULTS FOR OTHER FERROMAGNETS

In this section we go beyond the standard Ising ferromagnet by eliminating both the restriction to $\pm 1$ Ising variables and the restriction to nearest-neighbor interactions.

Let $\{\phi_x(\beta): x \in \mathbb{Z}^d\}$ be a family of random variables whose joint distribution is the infinite-volume limit with free boundary conditions of the finite-volume Gibbs distribution proportional to

$$\left[ \exp\left( \frac{\beta}{2} \sum_{x,y} J_{xy} \phi_x \phi_y \right) \right] \prod_x \rho(d\phi_x) \tag{5.1}$$

Here $\rho$ is an even Borel probability measure on $\mathbb{R}$ with $\rho \neq \delta_0$, the point measure at the origin, and such that

$$\int_{\mathbb{R}} \exp(b\phi^2)\, \rho(d\phi) < \infty \qquad \text{for all} \quad b > 0 \tag{5.2}$$

The couplings $J_{xy}$ satisfy

$$J_{xy} = J_{y-x} \geqslant 0 \qquad \text{with} \quad 0 < \sum_{x \in \mathbb{Z}^d} J_x < \infty \tag{5.3}$$

We continue to define $M_n(\beta)$, $E_n(\beta)$, $K_n(\beta)$, $L_n(\beta)$, $X_n$, and $Y_n$ as in Section 2, but with $\{\sigma_x(\beta)\}$ replaced by $\{\phi_x(\beta)\}$.

The class of measures $\rho$ on $\mathbb{R}$ to which our results will apply is the class $\mathfrak{F}$ basically introduced in refs. 16 and 33 and discussed further in refs. 29 and 5 (but see the remark following Lemma 6 below). A measure is in $\mathfrak{F}$ if it is the distribution of a finite positive linear combination of spins from some finite $\pm 1$ Ising system with zero external field and pair ferromagnetic interactions or if it is the weak limit of a sequence of such distributions. Examples of measures in $\mathfrak{F}$ are the spin-$n/2$ measures, $\rho = (\delta_n + \delta_{n-2} + \cdots + \delta_{-n})/(n+1)$ for $n = 1, 2,...,$[16] and the $\phi^4$ measures, const $\cdot \exp(-\lambda\phi^4 + b\phi^2)\, d\phi$ with $\lambda > 0, b \in \mathbb{R}$.[33]

The next theorem is our extension of Theorem 1. In a typical application, $\bar{\beta}$ would be the critical parameter $\beta_c$ and the hypothesis that the bubble quantity is finite at $\beta_c$ would only be valid if $d$ exceeds the upper critical dimension for the given couplings $J_x$. Two specific situations in which the theorem can be applied in that way are nearest-neighbor $\phi^4$ models with $d > 4$[2] and models with couplings of the form $J_x = (1 + |x^1| \cdots + |x^d|)^{-\tau}$, where $1 \leqslant d \leqslant 4$ and $d < \tau < 3d/2$.[4]

**Theorem 5.** Let $\bar{\beta} \in [0, \infty)$ be such that

$$\sum_{x \in \mathbb{Z}^d} \langle \phi_0(\bar{\beta})\, \phi_x(\bar{\beta}) \rangle^2 < \infty \tag{5.4}$$

and let $\beta_n$ be any sequence in $[0, \bar{\beta}]$. Then, as $n \to \infty$, $(X_n, Y_n)$ converges in distribution to a pair of independent standard Gaussian random variables.

*Proof.* Since the proof is much the same as for Theorem 1, we will restrict ourselves to discussing the major points which are different. The first difference is that the bound (3.4) on $-u_{4,n}$ from ref. 2 is replaced by the related bound from ref. 5 (see also ref. 13 for some minor corrections to ref. 5), which yields [in place of (3.6)]

$$\begin{aligned} 0 \leqslant -U_{4,n} \leqslant [K_n(\beta_n)]^{-2} \Bigg\{ & 2A_n^2 \sum_{x, x' \in \Lambda_n} \sum_{y \in \mathbb{Z}^d} G_n(x-y)\, G_n(x'-y) \\ & + (C_1 A_n + C_2) \sum_{x, x', y \in \Lambda_n} G_n(x-y)\, G_n(x'-y) \Bigg\} \end{aligned} \tag{5.5}$$

where $C_1$ and $C_2$ are combinatorial constants and

$$A_n = \left( \sum_{w \in \mathbb{Z}^d} \beta_n J_w \right) \sup_{w \in \mathbb{Z}^d} \sum_{x \in \Lambda_n} G_n(x-w) \tag{5.6}$$

This leads to an analogue of (3.11) with $2^{2d+1}$ replaced by $2^{2d+1}(\bar{\beta} \sum_w J_w)^2$, providing (3.7) is valid. The second difference with the proof of Theorem 1 is an alternate derivation of (3.7) which does not require any monotonicity of $G_n(x)$ in the coordinates of $x$; this is given in Lemma 6 below. The final difference, which comes in the proof of Proposition 3, is to replace the proof of continuity in $\beta$ of

$Q(\beta) \equiv \langle \phi_{x_1}(\beta) \cdots \phi_{x_{2n}}(\beta) \rangle$ for $\beta \in [0, \bar{\beta}]$ from one relying on the arguments of refs. 23 and 24 to one which gives an *a priori* uniform bound on $dQ/d\beta$ (or more accurately on $\Delta Q/\Delta \beta$) by using the type of argument applied to $d\langle M_n^{2k} \rangle_r/dr$ in the proof of Proposition 4 above. The resulting bound, for $0 \leqslant \beta_1 \leqslant \beta_2 \leqslant \bar{\beta}$, is

$$
\begin{aligned}
0 &\leqslant Q(\beta_2) - Q(\beta_1) \\
&\leqslant \text{const} \cdot \max_{i,j} \int_{\beta_1}^{\beta_2} \sum_{x, y \in \mathbb{Z}^d} J_{y-x} \langle \phi_{x_i}(\beta)\, \phi_x(\beta) \rangle \langle \phi_{x_j}(\beta)\, \phi_y(\beta) \rangle \, d\beta \\
&\leqslant \text{const} \cdot \left( \sum_w J_w \right) \int_{\beta_1}^{\beta_2} \sum_{x \in \mathbb{Z}^d} \langle \phi_0(\beta)\, \phi_x(\beta) \rangle^2 \, d\beta \\
&\leqslant \text{const} \cdot \left( \sum_w J_w \right) \left[ \sum_{x \in \mathbb{Z}^d} \langle \phi_0(\bar{\beta})\, \phi_x(\bar{\beta}) \rangle^2 \right] (\beta_2 - \beta_1)
\end{aligned}
$$

We leave further details of the derivation of this bound to the reader, except to note that identities involving expressions such as $dQ/d\beta$ should be understood as being evaluated before the infinite-volume limit is taken.

*Remark.* $\langle \phi_0(\beta)\, \phi_x(\beta) \rangle$ is always right-continuous in $\beta$ because it is nondecreasing and is obtained in the infinite-volume limit as the decreasing limit of continuous functions. It follows that the basic hypothesis (5.4) of Theorem 5 is implied by the slightly weaker assumption that

$$\lim_{\beta \uparrow \bar{\beta}} \sum_{x \in \mathbb{Z}^d} \langle \phi_0(\beta)\, \phi_x(\beta) \rangle^2 < \infty$$

We complete the proof of Theorem 5 by providing the previously promised Lemma 6. Its proof was suggested by M. Aizenman.

**Lemma 6.** Suppose $G(x)$ is a function on $\mathbb{Z}^d$ which is both pointwise nonnegative and positive semidefinite, i.e.,

$$\sum_{i,j=1}^{n} \bar{C}_i C_j G(x_j - x_i) \geqslant 0$$

for any $n$, any $x_1, ..., x_n \in \mathbb{Z}^d$, and any $C_1, ..., C_n \in \mathbb{C}$ (where $\bar{C}$ denotes the complex conjugate of $C$). Then for any $y \in \mathbb{Z}^d$,

$$\sum_{x \in \Lambda_n} G(x - y) \leqslant 2^d \, |\Lambda_n|^{-1} \sum_{x_1, x_2 \in \Lambda_n} G(x_2 - x_1) \tag{5.7}$$

*Proof.* Define $\tilde{I}_n(x)$ to be the indicator function of $\Lambda_n$ and $I_n(x)$ as

$$I_n(x) = |\Lambda_n|^{-1} \sum_{y \in \mathbb{Z}^d} \tilde{I}_n(x-y)\, \tilde{I}_n(y)$$

$$= \begin{cases} \prod_{j=1}^{d} (1 - |x^j|/(2n+1)) & \text{if} \quad |x^j| \leqslant 2n \text{ for each } j \\ 0 & \text{otherwise} \end{cases}$$

Then $I_n(x) \geqslant (1/2)^d$ for $x \in \Lambda_n$ and so

$$\sum_{x \in \Lambda_n} G(x-y) \leqslant 2^d \sum_{x \in \mathbb{Z}^d} I_n(x)\, G(x-y)$$

$$= 2^d \int_{[-\pi,\pi]^d} e^{+i(k,y)} \hat{I}_n(k)\, \hat{G}(dk) \tag{5.8}$$

where $\hat{G}(dk)$ is the finite positive measure on $[-\pi, \pi]^d$ such that

$$G(x) = (2\pi)^{-d/2} \int_{[-\pi,\pi]^d} e^{-i(k,x)} \hat{G}(dk) \tag{5.9}$$

(which exists by the positive semidefiniteness of $G$) and

$$\hat{I}_n(k) = (2\pi)^{-d/2} \sum_{x \in \mathbb{Z}^d} I_n(x)\, e^{i(k,x)}$$

$$= (2\pi)^{-d/2}\, |\Lambda_n|^{-1} \left[ \sum_{x \in \mathbb{Z}^d} \tilde{I}_n(x)\, e^{i(k,x)} \right]^2 \geqslant 0 \tag{5.10}$$

Since $\hat{I}_n$ and $\hat{G}$ are both positive, it follows from (5.8) that

$$\sum_{x \in \Lambda_n} G(x-y) \leqslant 2^d \int_{[-\pi,\pi]^d} \hat{I}_n(k)\, \hat{G}(dk)$$

$$= 2^d \sum_{x \in \mathbb{Z}^d} I_n(x)\, G(x)$$

$$= 2^d\, |\Lambda_n|^{-1} \sum_{x_1, x_2 \in \mathbb{Z}^d} \tilde{I}_n(-x_1)\, \tilde{I}_n(x_2)\, G(x_2 - x_1)$$

which gives (5.7) as desired.

*Remark.* Theorem 5 is valid for other single-site measures $\rho$ than those in the class $\mathfrak{F}$ defined above. It suffices, for example, that $\rho$ satisfy the following two conditions:

(i) $\int_{-\infty}^{+\infty} \exp(z\phi)\, \rho(d\phi)$ has only pure imaginary zeros.

(ii) $\rho(d\phi) = g(\phi^2)\, d\phi$ with $g$ log-concave.

Condition (i) implies that also the zeros of $\langle \exp(zX_n) \rangle$ are all pure imaginary[27] [yielding (3.2)] while condition (ii), introduced in ref. 8, implies both an inequality on $U_{4,n}$ similar to (5.5)[6,20] and truncated Gaussian inequalities such as (4.24). An example of a measure satisfying (i) and (ii) is

$$\rho(d\phi) = \text{const} \cdot (1 + \phi^2) \exp(-\lambda\phi^4 + b\phi^2)\, d\phi$$

with $\lambda > 0$ and $b \in \mathbb{R}$; this $\rho$ is not in $\mathfrak{F}$ for small $\lambda$ and large negative $b$ because it does not yield a GHS inequality.[11] Theorem 5 can also be extended by allowing $n$ dependence in the couplings $J_x = J_x^{(n)}$ and/or in the single-site measure $\rho = \rho^{(n)}$. In particular, we note that for convergence of $X_n$ to a standard normal random variable, it suffices [see (3.14), (5.5) and (5.7)] that

$$\left( \sum_{w \in \mathbb{Z}^d} \beta_n J_w^{(n)} \right)^2 |\Lambda_n|^{-1} \sum_{x \in \Lambda_{2n}} \int_{[-\pi,\pi]^d} e^{-i(k,x)} [\hat{G}_n(k)]^2 \, dk \to 0$$

as $n \to \infty$. By the infrared bounds of ref. 14, this will be the case, for example, in a nearest neighbor model with $d > 4$ provided each $\beta^{(n)}$ is in the single-phase region for the corresponding $\rho^{(n)}$.

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